

# The Origin of Spatial Intermittency in the Galaxy Distribution

Peter Coles

*School of Physics & Astronomy, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom*

1 February 2008

## ABSTRACT

The dynamical equations describing the evolution of a self-gravitating fluid can be rewritten in the form of a Schrödinger equation coupled to a Poisson equation determining the gravitational potential. This approach has a number of interesting features, many of which were pointed out in a seminal paper by Widrow & Kaiser (1993). In particular we show that this approach yields an elegant reformulation of an idea due to Jones (1999) concerning the origin of lognormal intermittency in the galaxy distribution.

**Key words:** Cosmology: theory – galaxies: clustering – large-scale structure of the Universe.

## 1 INTRODUCTION

Galaxy redshift surveys such as the 2dF Galaxy redshift survey and the Sloan Digital Sky Survey are poised to yield a copious harvest of statistical information about the distribution and dynamical properties of the large-scale structure of the Universe. At the same time relatively few statistical properties of the structure revealed by these observational programs are understood in quantitative detail using analytic methods.

One particular facet that has received some attention over the years has been the property that the one-point probability distribution of density fluctuations  $p(\rho)$  appears to have a roughly lognormal form, i.e.  $\log \rho$  has a roughly normal distribution (Coles & Jones 1991). It is now established that this property has an interesting connection with the scaling properties of moments of the probability of the distribution. Taking a generic random variable  $X$ , such that the distribution of  $X$  within cells of side  $L$  is denoted  $p(X; L)$ , then the  $q$ -th moment at a given value of  $L$  is said to display scaling if

$$\langle X^q \rangle_L = \int p(X; L) X^q \propto L^{\mu(q)}. \quad (1)$$

This means that different powers  $q$  of the density field vary as a different power of the coarse-graining scale  $L$ . The function  $\mu(q)$  is called the *intermittency exponent*, and it can be extracted from observations. Jones, Coles & Martinez (1992) showed that observations suggest a roughly quadratic dependence of  $\mu(q)$  upon  $q$  and that this is related to the underlying near-lognormal form of the density fluctuations. A set displaying the form (1) is usually termed a multifractal; see Paladin & Vulpiani (1987) for general discussion.

The term *intermittency* was coined to describe properties of stochastic processes described by high skewed probability distributions with very slowly decaying tails. The lognormal is a prime example, used in a pioneering paper by Kolmogorov (1962). In the time domain it refers to processes involving extended quiescent periods interrupted by bursts of intense activity. In the spatial domain, intermittent processes are ones in which isolated regions of high density are separated by large voids. Although in a qualitative sense the application of the concept of intermittency to large-scale structure seems plausible, a quantitative description of how it arises is not easy to obtain. Drawing on ideas discussed by Zel'dovich et al. (1985, 1987), Jones (1999) suggested an analytical model for the cosmological context. It is the purpose of this paper to describe an alternative formulation of the Jones (1999) model and substantially strengthens the physical understanding of this model. This also provides an opportunity to advocate the wider use of an alternative formulation of the gravitational instability scenario discussed by Widrow & Kaiser (1993).

## 2 THE FLUID APPROACH TO STRUCTURE FORMATION

### 2.1 Basics

In order to understand how the intermittent form of large-scale structure arises, it is best to begin with the standard fluid-based approach to structure growth. In the standard treatment of the Jeans Instability one begins with the dynamical equations governing the behaviour of a self-gravitating fluid. These are: the *Euler equation*

$$\frac{\partial(\mathbf{v})}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \nabla\phi = 0 ; \quad (2)$$

the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0 \quad (3)$$

expressing the conservation of matter; and the Poisson equation

$$\nabla^2\phi = 4\pi G\rho, \quad (4)$$

describing Newtonian gravity.

## 2.2 The Cosmological Setting

If the length scale of the perturbations is smaller than the effective cosmological horizon  $d_H = c/H_0$ , a Newtonian treatment of cosmic structure formation is still expected to be valid in expanding world models. In an expanding cosmological background, the Newtonian equations governing the motion of gravitating particles can be written in terms of

$$\mathbf{x} \equiv \mathbf{r}/a(t) \quad (5)$$

(the comoving spatial coordinate, which is fixed for observers moving with the Hubble expansion),

$$\mathbf{v} \equiv \dot{\mathbf{r}} - H\mathbf{r} = a\dot{\mathbf{x}} \quad (6)$$

(the peculiar velocity field, representing departures of the matter motion from pure Hubble expansion),  $\rho(\mathbf{x}, t)$  (the matter density), and  $\phi(\mathbf{x}, t)$  (the peculiar Newtonian gravitational potential, i.e. the fluctuations in potential with respect to the homogeneous background) determined by the Poisson equation in the form

$$\nabla_{\mathbf{x}}^2\phi = 4\pi G a^2(\rho - \rho_0) = 4\pi G a^2\rho_0\delta. \quad (7)$$

In this equation and the following the suffix on  $\nabla_x$  indicates derivatives with respect to the new comoving coordinates. Here  $\rho_0$  is the mean background density, and

$$\delta \equiv \frac{\rho - \rho_0}{\rho_0} \quad (8)$$

is the *density contrast*. Using these variables the Euler equation becomes

$$\frac{\partial(a\mathbf{v})}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{1}{\rho}\nabla_{\mathbf{x}}p - \nabla_{\mathbf{x}}\phi. \quad (9)$$

The first term on the right-hand-side of equation (9) arises from pressure gradients, and is neglected in dust-dominated cosmologies. Pressure effects may nevertheless be important in the (collisional) baryonic component of the mass distribution when nonlinear structures eventually form. The second term on the right-hand side of equation (9) is the peculiar gravitational force, which can be written in terms of  $\mathbf{g} = -\nabla_{\mathbf{x}}\phi/a$ , the peculiar gravitational acceleration of the fluid element. If the velocity flow is irrotational,  $\mathbf{v}$  can be rewritten in terms of a velocity potential  $\phi_v$ :

$$\mathbf{v} = -\nabla_{\mathbf{x}}\phi_v/a. \quad (10)$$

This is expected to be the case in the cosmological setting because (a) there are no sources of vorticity in these equations and (b) vortical perturbation modes decay with the expansion. Next we have the revised continuity equation:

$$\frac{\partial\rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_{\mathbf{x}}(\rho\mathbf{v}) = 0, \quad (11)$$

## 2.3 The Nonlinear regime

The strength of the fluid approach is that it furnishes a relatively simple perturbative treatment for the early stages of structure formation, wherein the density fluctuations are small. Perturbation theory fails when nonlinearities develop, and unfortunately this it is in the nonlinear regime where scaling and intermittency arise. It is important to stress, however, that the fluid treatment is intrinsically approximate anyway. A fuller treatment of the problem requires a solution of the Boltzmann equation for the full phase-space distribution of the system  $f(\mathbf{x}, \mathbf{v}, t)$  coupled to the Poisson equation (4) that determines the gravitational potential. In cases where the matter component is collisionless, the Boltzmann equation takes the form of a Vlasov equation:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^3 \left( \frac{\partial\phi}{\partial x_i} \frac{\partial f}{\partial v_i} - v_i \frac{\partial f}{\partial x_i} \right). \quad (12)$$

The fluid approach can only describe cold material where the velocity dispersion of particles is negligible. But even if the dark matter is cold, there may be hot components of baryonic material whose behaviour needs also to be understood. Moreover, the fluid approach assumes the existence of a single fluid velocity at every spatial position. It therefore fails when orbits cross and multi-streaming generates a range of particle velocities through a given point. While the existence and location of the caustics that occur and shell crossing can be determined using the Zel'dovich (1970) approximation (Coles, Melott & Shandarin 1993), the dynamics and distribution of matter after shell-crossing is not described in this approach. Attempts to understand properties of non-linear structure using the fluid model therefore resort to further approximations (Sahni & Coles 1995) to extend the approach beyond shell-crossing. Even the numerical N-body methods that many regard as the ultimate approach to this kind of problem are also approximate, corresponding to a particular type of Monte Carlo integration of the Vlasov equation.

## 2.4 The Jones Model

It is within the overall framework of the fluid model that Jones (1999) sought to understand the observed intermittency of the large-scale structure of the Universe. Using the velocity potential introduced above, he first introduces an effective Bernoulli equation for the flow:

$$\frac{\partial\phi_v}{\partial t} - \frac{(\nabla\phi_v)^2}{2a^2} = \phi, \quad (13)$$

where  $\phi$  is the actual gravitational potential. This equation neglects terms involving pressure gradients as mentioned above. To cope with shell-crossing events, Jones (1999) introduces an artificial viscosity  $\nu$  by adding a term to the right-hand-side of this equation:

$$\frac{\partial\phi_v}{\partial t} - \frac{(\nabla\phi_v)^2}{2a^2} = \phi + \frac{\nu}{a^2}\nabla^2\phi_v. \quad (14)$$

The viscosity is introduced to prevent the particle flow from entering the multi-stream region by causing particles to stick together at shell-crossing. This is also used in an approach called the adhesion approximation (Gurbatov,

Saichev & Shandarin 1989). Using the Hopf-Cole transformation  $\phi_v = -2\nu \log \varphi$  and defining a scaled gravitational potential via  $\phi = 2\nu\epsilon$  we can write the Bernoulli equation as

$$\frac{\partial \varphi}{\partial t} = \nu \nabla^2 \varphi + \epsilon(\mathbf{x})\varphi. \quad (15)$$

This is called the random heat equation, because of the existence of the spatially-fluctuating potential term  $\epsilon(\mathbf{x})$ . The gravitational potential changes very slowly even during nonlinear evolution (Brainerd, Scherrer & Villumsen 1993; Bagla & Padmanabhan 1994), so Jones (1999) assumes that it can be taken as constant and to be Gaussian distributed. An approximate scaling solution to (15) can then be found using a path integral adapted from that normally used in quantum physics (Feynman & Hibbs 1965); see below for more details. In this approximation, the function  $\varphi$  has a lognormal distribution (Coles & Jones 1991). We refer the reader to Jones (1999) for details; see also Zel'dovich et al. (1985, 1987).

This model is one of the few attempts that have been made to understand the non-linear behaviour of the matter distribution using analytic methods. Although not rigorous it surely captures the essential factors involved. It does, however, suffer from a number of shortcomings. First, the approach does not follow material beyond the shell-crossing stage. Second, the viscosity  $\nu$  that is needed does not have properties that are very realistic physically: it can depend neither on the density  $\rho$  nor the position  $\mathbf{x}$ . Moreover, in the final analysis Jones (1999) takes the limit  $\nu \rightarrow 0$ , so it cancels out anyway. One is tempted to speculate that its introduction may be unnecessary. Third, the function  $\varphi(\mathbf{x}, t)$  that emerges from equation (15) is not the desired density  $\rho(\mathbf{x}, t)$  nor does it bear a straightforward relation to the density. Finally, it is not clear how the motion of a collisional baryonic component can be modelled within this framework.

### 3 WAVE MECHANICS AND STRUCTURE FORMATION

A novel approach to the study of collisionless matter was suggested by Widrow & Kaiser (1993). They advocate the replacement of the Euler and continuity equations by a version of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + m\phi(\mathbf{x})\psi. \quad (16)$$

The equivalence between this and the fluid approaches has been known for some time; see Spiegel (1980) for historical comments. Originally the interest was to find a fluid interpretation of quantum mechanical effects, but in this context we shall use it to describe an entirely classical system. In this spirit, the constant  $\hbar$  is taken to be an adjustable parameter that controls the spatial resolution  $\lambda$  through a de Broglie relation  $\lambda = \hbar/mv$ . To accommodate gravity we need to couple equation (16) to the Poisson equation in the form

$$\nabla^2 \phi = 4\pi G \psi \psi^*. \quad (17)$$

It clearly follows that  $\rho = |\psi|^2$ ; the wavefunction  $\psi(\mathbf{x}, t)$  evidently complex. Moreover, if the matter distribution is initially cold then one can further construct a wavefunction

that also encodes the velocity part of phase space in its argument through the ansatz

$$\psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})} \exp[i\theta(\mathbf{x}/\hbar)], \quad (18)$$

where  $\nabla\theta(\mathbf{x}) = \mathbf{p}(\mathbf{x})$ , the local ‘momentum field’. This formalism thus yields an elegant description of both the density and velocity fields in a single function.

This formulation provides a useful complementary approach to many techniques, including  $N$ -body methods. It also provides a new light with which to study the Jones (1999) model. Widrow & Kaiser (1993) show using theoretical arguments and numerical simulations that this system allows accurate numerical evolution of the system beyond shell-crossing, so it does not have the ad hoc construction needed by the Jones (1999) model to remedy this.

Second, no artificial viscosity is required. Equations (15) and (16) are of the same form, apart from minor subtleties like the use of complex time coordinates. The potential term is easily understood in (16), and the wavefunction  $\psi$  now has a straightforward relationship to  $\rho$ . The upshot of this is that if one adopts the approximation of constant gravitational potential one can use the same path integral approach as described by Jones (1999). In a nutshell, given some initial wavefunction  $\psi(\mathbf{x}', t')$  one can determine the wavefunction at a subsequent time  $\psi(\mathbf{x}, t)$  using

$$\psi(\mathbf{x}, t) = \int K(\mathbf{x}, t; \mathbf{x}', t') \psi(\mathbf{x}', t') d^3x', \quad (19)$$

where the function  $K(\mathbf{x}, t; \mathbf{x}', t')$  involves a sum over all paths  $\Gamma$  connecting the initial and final states with  $t > t'$ :

$$K(\mathbf{x}, t; \mathbf{x}', t') = \int \mathcal{D}\Gamma \exp[iS(\Gamma)/\hbar], \quad (20)$$

where  $\mathcal{D}$  is an appropriate measure on the set of classical space-time trajectories. For a particle moving in a potential  $V(\mathbf{x}, t) = m\phi(\mathbf{x}, t)$  the action  $S$  for a given path  $\Gamma$  is given by

$$S(\Gamma) = \int_{\Gamma} \left[ \frac{1}{2} m \dot{x}^2 - m\phi(\mathbf{x}) \right] dt. \quad (21)$$

Note the presence of the Gaussian field in equation (21) and hence in the exponential of the integrand on the right-hand-side of equation (20). To get an approximate solution to this system we can follow the same reasoning as Zeldovich et al. (1985, 1987) and Jones (1999), ignoring time-varying terms, using the Gaussian properties and counting the dominant contributions to the path integral to deduce that the integral produces a solution of lognormal form. This part of the argument is identical to that advanced by Jones, except that the solution is for  $\psi$  rather than  $\varphi$  and since  $\rho$  is  $|\psi|^2$  then one directly obtains a lognormal distribution for the desired density  $\rho(\mathbf{x}, t)$ .

It should be stressed that, although the present approach clearly provides a more elegant formulation of the problem, the deduction of lognormality remains approximate; the lognormal is not the exact solution to the system to either Jones’ equation (15) or the present equation (16). How accurately this approximate form applies is open to doubt and will have to be checked by full numerical solutions. Interestingly, however, it is known to apply quite accurately in quantum systems such as disordered mesoscopic electron configurations (Janssen 1998). As mentioned above,

the Schrodinger approach yields a wavefunction  $\psi$  which is directly related to the particle density  $\rho$  via  $\rho = |\psi|^2$ . Such quantum systems also display lognormal scaling for properties such as the conductance, which depends on  $|\psi|^4$ . It is a property of the lognormal distribution that if a random variable  $X$  is lognormal, then so is  $X^n$ . In such systems the role of the gravitational potential  $\phi$  is played by a potential that describes the disorder of a solid, perhaps caused by the presence of defects. Such systems display *localisation* at low temperature which is similar in some ways to the original idea of Anderson location (Anderson 1958). The formation of strongly non-linear structures by gravity is thus directly analogous to the generation of localised wavefunctions in condensed matter systems.

Finally, and perhaps most promisingly for future work, the equation (16) offers a relatively straightforward way of modelling the behaviour of collisional material. The addition to the potential of a term of the form  $\alpha|\psi|^2$  (with  $\alpha$  an appropriately-chosen constant), converts the original equation (16) into a nonlinear Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + m\phi(\mathbf{x})\psi + \alpha|\psi|^2\psi \quad (22)$$

(Sulem & Sulem 1999). This equation is now equivalent to those that describe the flow of a barotropic fluid; see Spiegel (1980). This system can therefore be used to model pressure effects, which are otherwise only handled effectively using numerical methods such as smoothed-particle hydrodynamical approximations (e.g. Monaghan 1992). In the context of quantum systems, the nonlinear term is used to describe the formation of Bose-Einstein condensates (e.g. Choi & Niu 1999 and reference therein).

## 4 DISCUSSION

In this short paper I have sketched out an approach to the study evolving cosmological density fluctuations that relies on a transformation of the Vlasov-Poisson system into a Schrödinger-Poisson system. The transformation is not a new idea, but despite the efforts of Widrow & Kaiser (1993) it does not seem to be well known in the astronomical community. The immediate advantage of this new formalism is that it yields a rather more convincing approach to understanding the origin of spatial intermittency and approximate lognormality in the galaxy distribution than that offered by Jones (1999). It also makes a connection in the underlying physics with other systems that display similar phenomena.

On the other hand, one must be aware of the approximations also inherent in the present approach. The Schrödinger equation is not exact, and its usefulness as an approximate tool is restricted by a number of conditions outlined by Widrow & Kaiser (1993); see also Spiegel (1980). Furthermore, the lognormal solution of the system is a further approximation and may not be valid especially in the strongly-fluctuating limit. Although it neatly bypasses some of the problems inherent in the Jones (1999) model, the nonlinear wave equation is by no means easy to solve in general situations. Numerical methods will still have to be employed to understanding other aspects of the evolution of cosmic structure within this framework as indeed they are in other branches of physics.

One particular issue worth exploring using this approach is to understand the limits of the approach in strongly non-linear situations. As it stands, the justification for the lognormal approximation arises from the weakly non-linear behaviour of collisionless matter moving in an almost constant potential field. Taking into account the expansion of the Universe, the changing gravitational potential, and the possible effects of matter pressure within in the action formalism described in Section 3 may well reveal that a different form of hierarchical scaling pertains in the strongly non-linear regime. One aspect of this is that the hierarchy of correlation functions that describe a lognormal distribution display Kirkwood (1935) scaling, while it appears from numerical  $N$ -body simulations that cosmological fluctuations display a different hierarchical form. For a discussion of the relationship between lognormal and hierarchical scaling, see Coles & Frenk (1991).

Apart from the particular topic of intermittency, this paper is also intended to stimulate interest in this general approach. Many techniques exist for studying the wave mechanics of disordered systems, such as the renormalization group and path-integral methods, few of which are used by cosmologists working in this area. It is to be hoped that the introduction of some of these methods may allow better physical insights into the behaviour of non-linear structure formation than can be found using brute-force  $N$ -body techniques.

## REFERENCES

- Anderson P.W., 1958, Phys. Rev., 109, 1492
- Bagla J.S., Padmanabhan T., 1994, MNRAS, 266, 227
- Brainerd T.G., Scherrer R.J., Villumsen J.V., 1993, ApJ, 418, 570
- Choi D.-I., Niu Q., 1999, Phys. Rev. Lett., 82, 2022
- Coles P., Frenk C.S., 1991, MNRAS, 253, 727
- Coles P., Jones B.J.T., 1991, MNRAS, 248, 1
- Coles P., Melott A.L. & Shandarin S.F., 1993, MNRAS, 260, 765
- Feynman R.P., Hibbs A.R., 1965, Quantum Mechanics and Path Integrals. McGraw Hill, New York
- Gurbatov S.N., Saichev A.I., Shandarin S.F., 1989, MNRAS, 236, 385
- Janssen M., 1998, Physics Reports, 295, 1
- Jones B.J.T., 1999, MNRAS, 307, 376
- Jones B.J.T., Coles P., Martinez V.J., 1992, MNRAS, 259, 246
- Kirkwood J.C., 1935, J. Chem. Phys., 3, 300
- Kolmogorov A.N., 1962, J. Fluid Mech., 13, 82
- Monaghan J.J., 1992, ARA& A, 30, 543
- Paladin G., Vulpiani A., 1987, Physics Reports, 156, 147
- Sahni V., Coles P., 1995, Physics Reports, 262, 1
- Shandarin S.F., Zel'dovich Ya. B., 1989, Rev. Mod. Phys., 61, 185
- Spiegel E.A., 1980, Physica, 1D, 236
- Sulem C., Sulem P.-L., 1999, The Nonlinear Schrodinger Equation. Self focusing and Wave Collapse. Springer, New York
- Widrow L.M., Kaiser N., 1993, ApJ, 416, L71
- Zel'dovich Ya.B., 1970, A& A, 5, 84
- Zel'dovich Ya. B., Molchanov S.A., Ruzmaikin A.A., Sokolov D.D., 1985, Sov. Phys. JETP, 62, 1188
- Zel'dovich Ya. B., Molchanov S.A., Ruzmaikin A.A., Sokolov D.D., 1987, Sov. Phys. Usp., 152, 3